

# All Ramsey numbers for brooms in graphs

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## Abstract

For  $k, \ell \geq 1$ , a broom  $B_{k,\ell}$  is a tree on  $n = k + \ell$  vertices obtained by connecting the central vertex of a star  $K_{1,k}$  with an end-vertex of a path on  $\ell - 1$  vertices. As  $B_{n-2,2}$  is a star and  $B_{1,n-1}$  is a path, their Ramsey number have been determined among rarely known  $R(T_n)$  of trees  $T_n$  of order  $n$ . Erdős, Faudree, Rousseau and Schelp determined the value of  $R(B_{k,\ell})$  for  $\ell \geq 2k \geq 2$ . We shall determine all other  $R(B_{k,\ell})$  in this paper, which says that, for fixed  $n$ ,  $R(B_{n-\ell,\ell})$  decreases first on  $1 \leq \ell \leq 2n/3$  from  $2n - 2$  or  $2n - 3$  to  $\lceil \frac{4n}{3} \rceil - 1$ , and then it increases on  $2n/3 < \ell \leq n$  from  $\lceil \frac{4n}{3} \rceil - 1$  to  $\lfloor \frac{3n}{2} \rfloor - 1$ . Hence  $R(B_{n-\ell,\ell})$  may attain the maximum and minimum values of  $R(T_n)$  as  $\ell$  varies.

**Keywords:** Ramsey number; Tree; Broom

## 1 Introduction

Given a graph  $G$ , the *Ramsey number*  $R(G)$  is the smallest integer  $N$  such that every red-blue coloring of the edges of  $K_N$  contains a monochromatic  $G$ . Let  $T_n$  be a tree of order  $n$ . Finding  $R(T_n)$  for an arbitrary  $T_n$  is a difficult unsolved problem in Ramsey theory. Most works focus on improving the known bounds, see [10]. Erdős and Sós conjectured that if a graph  $G$  has average degree greater than  $n - 1$ , then  $G$  contains every tree of  $n$  edges, which implies that  $R(T_n) \leq 2n - 2$  for  $n \geq 2$ . A result of Erdős, Faudree, Rousseau and Schelp in [4] yields

$$r(T_n) \geq \left\lceil \frac{4n}{3} \right\rceil - 1, \quad (1)$$

under (2) by minimizing the lower bound with  $b = 2a$ , and the lower bound can be attained by some brooms. For  $k, \ell \geq 1$ , a broom  $B_{k,\ell}$  is a tree on  $k + \ell$  vertices obtained by connecting the central vertex of a star  $K_{1,k}$  with an end-vertex of a path on  $\ell - 1$

vertices. Thus  $B_{k,1} = K_{1,k}$ ,  $B_{k,2} = K_{1,k+1}$  and  $B_{1,\ell} = P_{\ell+1}$ , where  $P_{\ell+1}$  is a path of order  $\ell + 1$ . They obtained the following result.

**Theorem 1.** ([4]) *Let  $k$  and  $\ell$  be integers with  $\ell \geq 2k \geq 2$  and  $n = k + \ell$ . Then*

$$R(B_{k,\ell}) = n + \left\lceil \frac{\ell}{2} \right\rceil - 1.$$

Thus  $R(B_{k,\ell}) = \lceil \frac{4n}{3} \rceil - 1$  for  $\ell \in \{2k, 2k+1, 2k+2\}$  and  $n = k + \ell$ , which attain the lower bound in (1). In this paper, we shall determine the values of  $R(B_{k,\ell})$  for  $1 \leq \ell \leq 2k - 1$ .

Note that when  $k$  is fixed and  $\ell$  is sufficient large,  $B_{k,\ell}$  is similar to a path  $P_n$ ; when  $\ell$  is fixed and  $k$  is sufficient large,  $B_{k,\ell}$  is similar to a star  $K_{1,n-1}$ . Among few known results of  $R(T_n)$ ,  $R(P_n)$  and  $R(K_{1,n-1})$  have been determined completely. For  $n \geq 2$ , the exact value of  $R(P_n)$  was determined in [7] as

$$R(P_n) = \left\lfloor \frac{3n}{2} \right\rfloor - 1,$$

and  $R(K_{1,n-1})$  was determined in [3] as

$$R(K_{1,n-1}) = \begin{cases} 2n - 3 & \text{if } n \text{ is odd,} \\ 2n - 2 & \text{otherwise.} \end{cases}$$

As  $B_{1,\ell} = P_{\ell+1}$ ,  $B_{k,1} = K_{1,k}$  and  $B_{k,2} = K_{1,k+1}$ , their Ramsey numbers can be determined by the above results. It was proved that  $R(B_{k,3}) = R(K_{1,k+1})$  in [2]. Thus we shall consider the case  $\ell \geq 4$  and  $k \geq 2$ .

**Theorem 2.** *Let  $k$  and  $\ell$  be integers with  $k \geq 2$  and  $n = k + \ell$ . Then*

$$R(B_{k,\ell}) = \begin{cases} n + \left\lceil \frac{\ell}{2} \right\rceil - 1 & \text{if } \ell \geq 2k - 1, \\ 2n - 2\left\lceil \frac{\ell}{2} \right\rceil - 1 & \text{if } 4 \leq \ell \leq 2k - 2. \end{cases}$$

*Remark.* Roughly speaking, for fixed  $n$ ,  $R(B_{n-\ell,\ell})$  decreases first on  $2 \leq \ell \leq \frac{2n-1}{3}$  from  $2n - 2$  or  $2n - 3$  to  $\lceil \frac{4n}{3} \rceil - 1$ , and then increases on  $\frac{2n-1}{3} < \ell \leq n$  from  $\lceil \frac{4n}{3} \rceil - 1$  to  $\lfloor \frac{3n}{2} \rfloor - 1$ . Hence  $R(B_{n-\ell,\ell})$  may attain the maximum and minimum values of  $R(T_n)$  when  $\ell$  varies, as it is believed that  $R(K_{1,n-1})$  is the maximum value of  $R(T_n)$ .

## 2 Proofs

For any red-blue edge-coloring of  $K_N$ , denote  $R$  and  $B$  be the induced red and blue subgraph, respectively, and  $N_R(x)$  and  $N_B(x)$  be the red neighborhood and blue neighborhood of  $x$ , respectively. Let  $N_R[x] = N_R(x) \cup \{x\}$ ,  $N_B[x] = N_B(x) \cup \{x\}$ ,  $\deg_R(x) = |N_R(x)|$ , and  $\deg_B = |N_B(x)|$ . For a graph  $G$  and disjoint subset  $A$  and  $D$ , denote by  $G(A)$  the subgraph of  $G$  induced by  $A$ , and  $G(A, D)$  the bipartite subgraph of  $G$  induced by  $A$  and  $D$ . If  $G$  is the red-blue edge-colored  $K_N$ , we write  $G_R(A) = G(A) \cap R$  and  $G_R(A, D) = G(A, D) \cap R$ . Notation not specifically mentioned will follow from [1]. We do not distinguish the vertex set and the graph when there is no danger of confusion.

Consider a tree  $T_n$  as a bipartite graph with two parts of size  $a$  and  $b$ , respectively, where  $a \leq b$ ,  $a + b = n$ . Observing that a red-blue edge-colored  $K_{2a+b-2}$  with  $R = K_{a-1} \cup K_{a+b-1}$  contains no monochromatic  $T_n$ , and a red-blue edge-colored  $K_{2b-2}$  with  $R = K_{b-1} \cup K_{b-1}$  contains no monochromatic  $T_n$ . We see that

$$R(T_n) \geq \max \left\{ 2a + b - 1, 2b - 1 \right\}, \quad (2)$$

where  $a$  and  $b$  are determined by  $T_n$ .

Note that  $B_{k,\ell}$  is a bipartite graphs on parts of sizes  $a = \lceil \frac{\ell}{2} \rceil$  and  $b = k + \lfloor \frac{\ell}{2} \rfloor$ , then

$$R(B_{k,\ell}) \geq \max \left\{ k + \left\lceil \frac{3\ell}{2} \right\rceil - 1, 2k + 2 \left\lfloor \frac{\ell}{2} \right\rfloor - 1 \right\}. \quad (3)$$

We shall prove the cases  $\ell = 2k - 1$  and  $\ell = 4$  in Theorem 2 via the following two lemmas.

**Lemma 3.** *Let  $k \geq 2$  be an integer. Then*

$$R(B_{k,2k-1}) = 4k - 2.$$

**Lemma 4.** *Let  $k \geq 2$  be an integer. Then*

$$R(B_{k,4}) = 2k + 3$$

In order to prove Lemma 3, we need two results from [9] and [6], respectively.

**Lemma 5.** ([9]) *Let  $G(A, D)$  be a bipartite graph on parts  $A$  and  $D$  with  $|A| = k$  and  $|D| = 2k - 2$  such that*

$$\min \{d(x) : x \in A\} \geq k.$$

*Then  $G(A, D)$  contains a cycle  $C_{2k}$ .*

**Lemma 6.** ([6])  *$R(C_{2k}) = 3k - 1$  for integer  $k \geq 3$*

*Proof of Lemma 3.* It is easy to see that  $R(B_{2,3}) = 6$ , we assume that  $k \geq 3$ . As the lower bound (3) implies  $R(B_{k,2k-1}) \geq 4k - 2$ , it suffices to show the opposite inequality.

Let  $G$  be a red-blue edge-colored  $K_{4k-2}$ . We shall show that  $G$  contains a monochromatic  $B_{k,2k-1}$ . By lemma 6,  $G$  contains a monochromatic cycle  $C_{2k}$ . Without loss of generality, we assume that this  $C_{2k}$  is blue and denote it by  $C_{2k}^{(B)}$ . Let  $D = G \setminus C_{2k}^{(B)}$ . Then  $|D| = 2k - 2$ . If there exists a vertex  $x \in C_{2k}^{(B)}$  such that  $|N_B(x) \cap D| \geq k - 1$ , then  $G$  contains a blue  $B_{k,2k-1}$ . We then assume that  $|N_B(x) \cap D| \leq k - 2$  for each  $x \in C_{2k}^{(B)}$ . The fact that

$$|N_B(x) \cap D| + |N_R(x) \cap D| = |D| = 2k - 2$$

implies that  $|N_R(x) \cap D| \geq k$  for each  $x \in C_{2k}^{(B)}$ , hence the number of red edges between  $C_{2k}^{(B)}$  and  $D$  is at least  $2k^2$ . So there exists a vertex  $u \in D$  such that

$$|N_R(u) \cap C_{2k}^{(B)}| \geq \frac{2k^2}{|D|} = \frac{2k^2}{2k - 2} \geq k + 1.$$

Let  $\{u_0, u_1, \dots, u_k\} \subseteq N_R(u) \cap C_{2k}^{(B)}$ , and  $A = C_{2k}^{(B)} \setminus \{u_1, u_2, \dots, u_k\}$ . Then  $|A| = k$ . Consider the bipartite graph  $G_R(A, D)$ . By Lemma 5, there is a red cycle  $C_{2k}$  between  $A$  and  $D$ . Denote by  $C_{2k}^{(R)}$  the red  $C_{2k}$ . Since  $C_{2k}^{(R)}$  contains  $A$  hence  $u_0$ , so the graph on  $\{u_1, u_2, \dots, u_k\}, u, u_0, C_{2k}^{(R)}$  induced by red edges contains a red  $B_{k, 2k-1}$ .

This completes the proof of Lemma 3.  $\square$

*Proof of Lemma 4.* The lower bound (3) implies  $R(B_{k,4}) \geq 2k + 3$ , and we shall show  $R(B_{k,4}) \leq 2k + 3$ . Let  $G$  be a red-blue edge-colored  $K_{2k+3}$ . Assume that  $G$  contains no monochromatic  $B_{k,4}$ .

If  $k = 2$ ,  $2k + 3 = 7$ . As  $R(P_5) = 6 < 7$ , we suppose that  $G$  contain a red  $P_5$ . Label the vertices of the path in order as  $\{x_1, x_2, \dots, x_5\}$ , denote another two vertices as  $y_1, y_2$ . Since there is no red  $B_{2,4}$ , all the edges between  $\{x_2, x_4\}$  and  $\{y_1, y_2\}$  are red. If there is red edge between  $\{x_1, x_5\}$  and  $\{y_1, y_2\}$ , say edge  $x_5y_2$  is red. Then edges  $x_5y_1, x_3y_1, x_3y_2$  are blue. Now  $\{x_2, x_3\}, y_2, x_4, y_1, x_5$  contain a blue  $B_{2,4}$ , a contradiction. If all the edges between  $\{x_1, x_5\}$  and  $\{y_1, y_2\}$  are blue, then  $\{x_1, x_2\}, y_2, x_4, y_1, x_5$  contain a blue  $B_{2,4}$ , a contradiction.

Now we consider the case  $k \geq 3$ . As  $R(K_{1,k+1}) < 2k + 3$ , we suppose that there is a blue star  $K_{1,k+1}$ , which is denoted by  $K_{1,k+1}^{(B)}$ . Let  $x$  be the center of  $K_{1,k+1}^{(B)}$ ,  $A = K_{1,k+1}^{(B)} \setminus \{x\}$  and  $D = G \setminus K_{1,k+1}^{(B)}$ . Then  $|A| = |D| = k + 1$ .

**Claim.**  $D$  induces a red  $K_{k+1}$ .

**Proof.** Suppose to the contrary, there is a blue edge  $uv$  in  $D$ . Since  $G$  contains no blue  $B_{k,4}$ , the edges between  $\{u, v\}$  and  $A$  are all red, and thus all the edges between  $\{u, v\}$  and  $D \setminus \{u, v\}$  are blue from the assumption that  $G$  contains no red  $B_{k,4}$ . Now consider the blue edges between  $\{u, v\}$  and  $A$ . With a similar analysis, we get that  $D$  induces a blue  $K_{k+1}$  and all edges between  $D$  and  $A$  are red.

Now, consider the adjacency between  $x$  and a vertex of  $D$ , say  $xu$ , no matter what the color of  $xu$  is, we have a monochromatic  $B_{k,4}$ , leading to a contradiction and the claim is proved.

Now  $D$  is a red  $K_{k+1}$ . If there exists a red edge  $xw$  with  $w \in D$ , then  $D \cup \{x\}$  induces a red  $K_{1,k+1}$  with center  $w$ . As  $A = V(G) \setminus (D \cup \{x\})$ , a similar analysis for the above claim tells us that  $A$  is a blue  $K_{k+1}$ . If the number of blue edges between  $A$  and  $D$  is at least  $k + 2$ , then there exists a vertex  $y \in A$  such that  $|N_B(y) \cap D| \geq 2$ . Now choose two vertices  $\{y_1, y_2\} \subseteq N_B(y) \cap D$  and two vertices  $\{a_1, a_2\} \subseteq A \setminus y$ , then  $(A \setminus \{a_1, a_2, y\}) \cup \{y_1, y_2\}, y, a_1, a_2, x$  contains a blue  $B_{k,4}$ , a contradiction. Thus assume to the contrary, there exists a vertex  $z \in D$  such that  $|N_R(z) \cap A| \geq \frac{(k+1)^2 - (k+1)}{|D|} = k \geq 2$ . If  $w, z$  are the same vertex, we can choose two vertices  $\{z_1, z_2\} \subseteq N_R(z) \cap A$  and three vertices  $\{d_1, d_2, d_3\} \subseteq D \setminus z$  for  $|D| = k + 1 \geq 4$ . Then  $(D \setminus \{d_1, d_2, d_3, z\}) \cup \{z_1, z_2, x\}, z, d_1, d_2, d_3$  contain a red  $B_{k,4}$ . If  $w, z$  are different, choose a vertex  $d_1 \in D \setminus \{z, w\}$ , then  $(D \setminus \{z, w, d_1\}) \cup \{z_1, z_2\}, z, d_1, w, x$  contain a red  $B_{k,4}$ , a contradiction.

Finally, assume that  $x$  is adjacent to  $D$  completely blue. Choose any set  $F \subseteq A \cup D$  such that  $|F| = k + 1$  and denote  $M = V(G) \setminus (F \cup x)$ . A similar analysis for the claim says that  $M$  is a red  $K_{k+1}$ . The choice of  $F$  tells us that  $A \cup D$  is a red  $K_{2k+2}$ , hence  $G$

contains a red  $B_{k,4}$ , which is a contradiction too.

This completes the proof of Lemma 4.  $\square$

**Lemma 7.** *For integers  $k, \ell, N$  with  $5 \leq \ell \leq 2k - 2$  and  $N \geq 2k + 2\lfloor \frac{\ell}{2} \rfloor - 1$ , let the edges of  $K_N$  be colored by two colors  $i \equiv 0, 1 \pmod{2}$ . Suppose  $i$  is a color and  $x$  is a vertex such that*

$$\deg_i(x) = \max_v \max\{\deg_i(v), \deg_{i+1}(v)\}.$$

*If there exist vertices  $y, z \subseteq N_{i+1}(x)$ , not necessarily distinct, satisfying*

1.  $|N_{i+1}(y) \cap N_i(x)| \geq k$ , and
2.  $\deg_{i+1}(z) \geq N - \ell$

*then  $G$  contains a monochromatic  $B_{k,\ell}$ .*

*Proof of Lemma 7.* Since  $|N_{i+1}(y) \cap N_i(x)| \geq k$ , we can choose a subset  $A$  in  $N_{i+1}(y) \cap N_i(x)$  such that  $|A| = k$ . Let  $H = G \setminus A$ , then  $|H| = N - k$ . Since  $R(C_{2t}) = 3t - 1$  for  $t \geq 3$ ,  $H$  contains a monochromatic  $C_{2t}$  in color  $j$ , denoted by  $C_{2t}^{(j)}$ , where

$$2t \geq 2 \left\lfloor \frac{N - k + 1}{3} \right\rfloor \geq 2 \left\lfloor \frac{k + 2\lfloor \ell/2 \rfloor}{3} \right\rfloor \geq \ell$$

for  $5 \leq \ell \leq 2k - 2$ . The choice of vertex  $x$  implies  $\deg_i(x) \geq \deg_{i+1}(z) \geq N - \ell$ , and thus

$$|N_i[x] \setminus A| + |C_{2t}^{(j)}| \geq |H| + 1, \quad |N_{i+1}[z] \setminus A| + |C_{2t}^{(j)}| \geq |H| + 1,$$

which implies that both  $N_i[x] \setminus A$  and  $N_{i+1}[z] \setminus A$  contain a vertex of  $C_{2t}^{(j)}$ .

*Case 1.*  $y = z$ . If  $j = i$ , namely,  $C_{2t}^{(j)} = C_{2t}^{(i)}$  is in color  $i$ , there is a monochromatic  $B_{k,\ell}$  in color  $i$  in  $A \cup \{x\} \cup C_{2t}^{(i)}$ , and otherwise  $j = i + 1$ , there exists a monochromatic  $B_{k,\ell}$  in color  $i + 1$  in  $A \cup \{y\} \cup C_{2t}^{(i+1)}$ .

*Case 2.*  $y \neq z$ . Similarly, we can find a monochromatic  $B_{k,\ell}$  either in  $A \cup \{x\} \cup C_{2t}^{(j)}$  or in  $A \cup \{y, x, z\} \cup C_{2t}^{(j)}$ .

This completes the proof of Lemma 7.  $\square$

The next two lemmas are results about the extremal edges in graph that contains no path  $P_t$ .

**Lemma 8.** ([5]) *Let  $t \geq 2$  be an integer, and  $G$  a graph of order  $N$  that contains no  $P_t$ . Let  $e(G)$  be the number of edges of  $G$ , then  $e(G) \leq \frac{(t-2)N}{2}$ .*

**Lemma 9.** ([8]) *Let  $G(X_B, X_R)$  be a bipartite graph on parts  $X_B$  and  $X_R$  with  $|X_R| \leq |X_B|$ . If  $G(X_B, X_R)$  contains no  $P_{2t}$  with  $2(t-1) \leq |X_R|$ , then*

$$e(G(X_B, X_R)) \leq (t-1) \left[ |X_B| + |X_R| - 2(t-1) \right].$$

*Proof of Theorem 2.* We may assume that  $5 \leq \ell \leq 2k - 2$  from Theorem 1, Lemma 3 and Lemma 4.

Set  $N = 2n - 2\lceil \frac{\ell}{2} \rceil - 1 = 2k + 2\lfloor \frac{\ell}{2} \rfloor - 1$ . Let  $G$  be a red-blue edge-colored  $K_N$ , and let  $R$  and  $B$  be the induced red and blue subgraph, respectively. Without loss of generality, we may assume that the maximal monochromatic degree of  $G$  is the maximum blue degree and  $x$  is a vertex such that

$$\deg_B(x) = \max_v \max\{\deg_B(v), \deg_R(v)\}.$$

To simplify the notation, we write  $X_B = N_B(x)$ ,  $X_R = N_R(x)$  and

$$t = \ell + k - |X_B| - 1.$$

The choice of  $x$  implies  $N_B(u) \cap X_B \neq \emptyset$  for each vertex  $u \in X_R$  as otherwise  $N_B[x] \subseteq N_R(u)$  and thus  $\deg_R(u) > \deg_B(x)$ , which is impossible. We shall separate the proof into three cases depending on  $|X_B|$ .

*Case 1.*  $|X_B| < k + \ell - 1$ , and either  $G(X_R)$  contains a blue  $P_t$ , denoted by  $P_t^{(B)}$ , or  $G(X_B, X_R)$  contains a blue  $P_{2t}$ , denoted  $P_{2t}^{(B)}$ .

In  $G(X_B \cup X_R)$ , let  $P^{(B)}$  be the longest blue path extended from  $P_t^{(B)}$  such that one of its end-vertices is in  $X_B$  if  $G(X_R)$  contains a blue  $P_t$ , or that from  $P_{2t}^{(B)}$  otherwise. If  $|P^{(B)}| \geq \ell - 1$ , then there exists a blue  $B_{k,\ell}$ . Thus we assume that  $|P^{(B)}| \leq \ell - 2$ , then  $P^{(B)}$  fails to contain at least  $|X_B| - (\ell - 2 - t) = k + 1$  vertices of  $X_B$ . Let  $y$  be the other end-vertex of  $P^{(B)}$ . Then  $|N_R(y) \cap X_B| \geq k + 1$ . The maximality of  $|P^{(B)}|$  implies  $|N_R(y)| \geq N - 1 - (\ell - 2) = N - \ell + 1$ , which and Lemma 7 imply that  $G$  contains a monochromatic  $B_{k,\ell}$ .

*Case 2.*  $|X_B| \geq k + \ell - 1$ .

Let  $P^{(B)}$  be the longest blue path in  $G(X_B \cup X_R)$  that has an end-vertex in  $X_B$ . A similar analysis in Case 1 implies that  $G$  contains a monochromatic  $B_{k,\ell}$ .

*Case 3.*  $|X_B| < k + \ell - 1$ , and neither  $G(X_R)$  contains a blue  $P_t$  nor  $G(X_B, X_R)$  contains a blue  $P_{2t}$ .

As  $t = \ell + k - |X_B| - 1 \geq 2$ , then  $|X_B| \leq \ell + k - 3$  and  $|X_R| = N - 1 - |X_B| \geq k$ .

Since  $G(X_R)$  contains no blue  $P_t$ , Lemma 8 implies  $e(G_B(X_R)) \leq (t - 2)|X_R|/2$ . The choice of  $x$  implies that the  $\min\{\deg_R(v), \deg_B(v)\} \geq |X_R|$  for each vertex  $v$  of  $G$ , and thus

$$e(G_B(X_B, X_R)) \geq |X_R| \cdot |X_R| - (t - 2)|X_R| = (|X_R| - t + 2)|X_R|.$$

Since  $G(X_B, X_R)$  contains no blue  $P_{2t}$ , Lemma 9 yields

$$e(G_B(X_B, X_R)) \leq M_B,$$

where

$$M_B = (t - 1) \left[ |X_B| + |X_R| - 2(t - 1) \right].$$

**Claim for Case 3.**  $G(X_B, X_R)$  has at most  $|X_R|(|X_B - k|) - 1$  blue edges.

**Proof.** Suppose opposite, then

$$e(G_B(X_B, X_R)) \geq |X_R| \cdot \max\{|X_R| - t + 2, |X_B| - k\} \geq m_B.$$

where

$$m_B = k(|X_R| - t + 2) + (|X_R| - k)(|X_B| - k).$$

Note that  $M_B$  and  $m_B$  are upper and lower bound of number of blue edges in  $G(X_B, X_R)$ , respectively, and thus  $M_B \geq m_B$ .

*Case 3.1.*  $\ell$  is even. In this subcase,  $|X_B| + |X_R| = 2k + \ell - 2$  and

$$\begin{aligned} t &= \ell + k - |X_B| - 1 = |X_R| - k + 1, \\ m_B &= k(k + 1) + (|X_R| - k)(|X_B| - k), \\ M_B &= (|X_R| - k) \left[ |X_B| + k - (|X_R| - k) \right] \end{aligned}$$

we have

$$m_B - M_B = k^2 - 2k(|X_R| - k) + (|X_R| - k)^2 + k = (2k - |X_R|)^2 + k > 0,$$

which is a contradiction.

*Case 3.2.*  $\ell$  is odd. In this subcase,  $|X_R| + |X_B| = 2k + \ell - 3$  and

$$\begin{aligned} t &= \ell + k - |X_B| - 1 = |X_R| - k + 2, \\ m_B &= k^2 + (|X_R| - k)(|X_B| - k), \\ M_B &= (|X_R| - k + 1) \left[ |X_B| + k - (|X_R| - k) - 2 \right]. \end{aligned}$$

We have

$$\begin{aligned} m_B - M_B &= (2k - |X_R|)^2 + 3|X_R| - |X_B| - 4k + 2 \\ &= (2k - |X_R|)(2k - |X_R| - 4) + 2k - \ell + 5. \end{aligned}$$

For  $\ell \leq 2k - 2$ , is odd, we get  $|X_R| \leq \lfloor \frac{N-1}{2} \rfloor \leq 2k - 3$ . If  $|X_R| \leq 2k - 4$ ,  $m_B - M_B \geq 2k - \ell + 5 > 0$ ; if  $\ell_2 = 2k - 3$ ,  $m_B - M_B = 2k - \ell + 2 > 0$ , a contradiction, hence the claim holds.

We now have

$$e(G_R(X_R)) \geq \binom{|X_R|}{2} - \frac{(t-2)|X_R|}{2} \geq \frac{(k-1)|X_R|}{2}.$$

and

$$e((G_R(X_B, X_R)) \geq |X_R| \cdot |X_B| - [|X_R|(|X_B| - k) - 1] = k|X_R| + 1,$$

Recall  $X_R = N_R(x)$ , and thus

$$\sum_{v \in X_R} |N_R(v)| \geq e(G_R(X_B, X_R)) + 2e(G_R(X_R)) + |X_R| = 2k|X_R| + 1.$$

Therefore, there exist  $y, z \subseteq X_R = N_R(x)$ , not necessarily distinct, such that  $|N_R(y) \cap N_B(x)| \geq k + 1$  and  $|N_R(z)| \geq 2k + 1 \geq N - \ell$ . Then, Lemma 7 implies that  $G$  contains a monochromatic  $B_{k,\ell}$ .

This completes the proof of Theorem 2.  $\square$

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